NOTES ON MODULATIONAL INTERACTION OF SECONDARY STRUCTURES, PART I

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1. Short introduction on disparate scale interaction

In this lecture, we will introduce one generic class of nonlinear precesses called disparate scale interaction. Plasma turbulence itself has several explicit distinct characteristic length scales. For instance, ion gyroradius ρ_i and electron gyroradius ρ_e in magnetized plasmas. The Debye length for collective oscillation, and skin depth for magnetic perturbation.

One reason for the disparate scales is that $m_e \ll m_i$, thus plasmas have different fluctuation properties, e.g. plasma waves vs. ion-acoustic waves.

Nonlinear dynamics: unstable modes couple stable modes with common scale length. For example, drift waves are unstable when $k_\parallel \ll k_\theta$, then nonlinear interaction within like-scale will increase $k_\parallel$, allowing energy transfer to strongly damped modes.

Disparate scale interaction is in contrast to Kolmogorov cascade in neutral fluids. In cascade, the kinetic energy is transferred from large scale L to micro-scale l_d where the kinetic energy dissipated. And there is no preferred scale between L and l_d.

High ω high k fluctuation (small) $\rightarrow$ low ω low k structure (large) by effective stress, and couple energy to large scale; Low ω low k structure (large) $\rightarrow$ high ω high k fluctuation (small) by refraction or strain field, as shown in Fig. 1.1

Disparate scale interaction is a typical process for the generation of large scale structure [Diamond et al. PPCF 2005]. Formation of large scale by turbulence is similar to 'inverse cascade' in fluid dynamics. But in disparate scale interaction, the energy transfer directly to long-wavelength structure from small scales, while

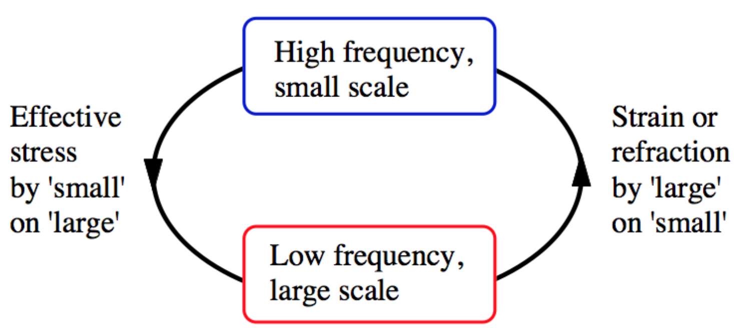

FIGURE 1.1. Interaction between small-scale fluctuations and large-scale ones.

in 'inverse cascade' case, the transfer occurs through a sequence of intermediate scales.

The disparate scale interactions has many examples. The simplest one is the interaction between Langmuir turbulence (plasma waves or plasmons) and ion-acoustic waves (phonons). In this case, the plasma waves form ponderomotive pressure field to ion-acoustic waves, and the density perturbation of ion-acoustic waves refracts plasma waves, so that the modulation of plasma waves grows.

The second example is the drift wave-zonal flow interaction in toroidal plasmas. In this case, small scale drift wave turbulence induces transport of momentum (Reynolds stress or vorticity flux), which amplifies the zonal flow shear. On the other hand, zonal flow shears stretch and tilt the drift wave packet. The coupling leads to an energy transfer from drift wave turbulence to zonal flows, which is an important nonlinear process for confinement of toroidal magnetized plasmas.

In the following contents, we will discuss these two examples.

2. Wave kinetic theory for Langmuir turbulence

In general situations, plasma waves are excited as Langmuir turbulence, and the ion-acoustic waves may also be a broad-band spectrum. The evolution of envelope of Langmuir turbulence then may be comparable to the ion-acoustic speed. The coupling between Langmuir turbulence and ion-acoustic waves is studied using quasi-particle approach here.

The Langmuir turbulence field is characterized by action density $N(\mathbf{k}, \mathbf{x}, t)$, i.e population density of waves.

$$N = \frac{E_k}{\omega_k}, \quad E_k = \frac{\partial}{\partial \omega}(\omega\epsilon)|_{\omega_k} \frac{\left|\tilde{\mathbf{E}}_k\right|^2}{8\pi}$$

where E_k is energy density, $\tilde{\mathbf{E}}_k$ is the electric field of plasma wave at wave number $\mathbf{k}$.

The ion-acoustic waves are described by the density and velocity perturbations $\tilde{n}$ and $\tilde{\mathbf{V}}$, both of which vary slowly compared to $\mathbf{k}$ and ω_k of Langmuir turbulence.

Under the condition of scale separation, the $N(\mathbf{k}, \mathbf{x}, t)$ is conserved along the trajectory. The wave kinetics equation for Langmuir turbulence under the influence of ion-acoustic waves is written as,

$$\frac{dN}{dt} = \frac{\partial N}{\partial t} + (\mathbf{v}_g + \tilde{\mathbf{V}}) \cdot \frac{\partial N}{\partial \mathbf{x}} - \frac{\partial}{\partial \mathbf{x}}(\omega_k + \mathbf{k} \cdot \tilde{\mathbf{V}}) \cdot \frac{\partial N}{\partial t} = 0 \tag{2.1}$$

the trajectories determined by eikonal equations,

$$\frac{d\mathbf{x}}{dt} = \frac{\partial \omega_k}{\partial \mathbf{x}} + \tilde{\mathbf{V}}, \quad \frac{d\mathbf{k}}{dt} = -\frac{\partial}{\partial \mathbf{x}}(\omega_k + \mathbf{k} \cdot \tilde{\mathbf{V}})$$

In the presence of long-scale perturbations, wave frequency is modified,

$$\omega_k = \omega_{k0} + \tilde{\omega}_k$$

ω_{k0} is given in the absence of acoustic waves, and the unperturbed orbit of quasi-particles is then,

$$\frac{d\mathbf{x}_0}{dt} = \frac{\partial \omega_{k0}}{\partial \mathbf{x}} = \mathbf{v}_g, \quad \frac{d\mathbf{k}_0}{dt} = -\frac{\partial \omega_{k0}}{\partial \mathbf{x}}$$

2.1. **Evolution of the Langmuir wave action density.** Set the action density of plasmons $N = \langle N \rangle + \tilde{N}$. We analyze the case that the Langmuir turbulence is homogeneous in unperturbed state, and the Doppler shift is smaller than effects of modulation of refraction, then the dispersion relation for plasma waves is

$$\omega^2 = \omega_{p0}^2 (1 + \frac{\tilde{n}}{n_0})$$

2.1.1. *Evolution of Langmuir wave energy density.* In interacting with acoustic waves, the action density N is conserved, then the change of energy density of plasma waves is

$$\frac{d}{dt} E_k = \frac{d}{dt}(\omega_k N) = N \frac{d}{dt}\omega_k$$

Noting the relations,

$$\frac{d}{dt}\omega_k = \frac{\partial \omega_k}{\partial \mathbf{k}} \cdot \frac{d\mathbf{k}}{dt} = \mathbf{v}_g \cdot (-\frac{\partial \omega_k}{\partial \mathbf{x}}) = -\mathbf{v}_g \cdot \frac{\partial}{\partial \mathbf{x}}(\frac{\omega_{p0}}{2}\frac{\tilde{n}}{n_0})$$

where we use

$$\frac{d\mathbf{k}}{dt} = -\frac{\partial \omega_k}{\partial \mathbf{x}} = -\frac{\omega_{p0}}{2n_0}\frac{\partial \tilde{n}}{\partial \mathbf{x}}$$

then one has,

$$\frac{dE_k}{dt} = -\frac{N\omega_{p0}}{2n_0}\mathbf{v}_g \cdot \frac{\partial \tilde{n}}{\partial \mathbf{x}}$$

Putting $N = \langle N \rangle + \tilde{N}$ into this relation, and 1st order terms vanishes in long time average, 2nd order terms survive,

$$\frac{d}{dt}\langle E_k \rangle = -\frac{\omega_{p0}}{2n_0}\mathbf{v}_g \cdot \langle \tilde{N} \frac{\partial \tilde{n}}{\partial \mathbf{x}} \rangle$$

This relation indicates that the change of plasma waves energy density, which is transferred to ion-acoustic waves, is given by the correlation $\langle \tilde{N} \frac{\partial \tilde{n}}{\partial \mathbf{x}} \rangle$.

2.1.2. *Wave kinetic equation of Langmuir action density.* Putting $N = \langle N \rangle + \tilde{N}$ into equation 2.1, yields the response of $\langle N \rangle$ and $\tilde{N}$ to ion-acoustic waves,

$$\frac{\partial \tilde{N}}{\partial t} + \mathbf{v}_g \cdot \frac{\partial \tilde{N}}{\partial \mathbf{x}} - \frac{\partial \omega_{k0}}{\partial \mathbf{x}} \cdot \frac{\partial \tilde{N}}{\partial \mathbf{k}} = \frac{\partial \tilde{\omega}_k}{\partial \mathbf{x}} \cdot \frac{\partial \langle N \rangle}{\partial \mathbf{k}} \tag{2.2}$$

$$\frac{\partial \langle N \rangle}{\partial t} = \frac{\partial}{\partial \mathbf{k}} \cdot \langle \frac{\partial \tilde{\omega}_k}{\partial \mathbf{x}} \tilde{N} \rangle$$

Here we neglect the $\tilde{\mathbf{V}}$ and $\mathbf{k} \cdot \tilde{\mathbf{V}}$, since the Doppler shift by the ion fluid motion is smaller than the effect of the modulation of refraction $\tilde{\omega}_k$.

2.2. **Linear response of distribution of quasi-particles.** Set the fluctuations to be

$$\tilde{n} = \sum_{q,\Omega} n_{q,\Omega} \exp(i\mathbf{q} \cdot \mathbf{x} - i\Omega t), \quad \tilde{N} = \sum_{q,\Omega} N_{q,\Omega} \exp(i\mathbf{q} \cdot \mathbf{x} - i\Omega t)$$

$\mathbf{q}$, Ω stand for the slow spatiotemporal variation associated with ion-acoustic waves. Then we can get response from equation 2.2,

$$N_{\mathbf{q},\Omega} = -\frac{\omega_{p0}}{\Omega - \mathbf{q} \cdot \mathbf{v}_g} \frac{n_{\mathbf{q},\Omega}}{2n_0} \mathbf{q} \cdot \frac{\partial \langle N \rangle}{\partial \mathbf{k}}$$

When the self-interaction of plasma waves is weaker than the decorrelation due to the wave dispersion, i.e. $\tau_{ac} < \tau_{tr}, \tau_c$, we can use quasi-linear theory to calculate the mean evolution of energy density,

$$\frac{d}{dt}\langle E_k\rangle = -D_N \frac{\partial\langle N\rangle}{\partial \mathbf{k}} \tag{2.3}$$

$$D_N = \left(\frac{\omega_{p0}}{2n_0}\right)^2 \Re \sum_{\mathbf{q},\Omega} |n_{\mathbf{q},\Omega}|^2 \frac{i}{\Omega - \mathbf{q}\cdot\mathbf{v}_g + i\gamma_N} \mathbf{v}_g \cdot \mathbf{q}\mathbf{q}\cdot \frac{\partial\langle N\rangle}{\partial \mathbf{k}}$$

$$\frac{i}{\Omega - \mathbf{q}\cdot\mathbf{v}_g + i\gamma_N} \to \pi\delta(\Omega - \mathbf{q}\cdot\mathbf{v}_g)$$

which is consistent with previous result.

Relation to wave-wave interaction

From Golden rule we know that

$$Rate \sim \delta(\omega_{k+q} - \omega_k - \omega_q) \approx \frac{i}{\omega_{k+q} - \omega_k - \omega_q}$$

In disparate scale interaction, we have $\mathbf{q} \ll \mathbf{k}$, then

$$\frac{i}{\omega_{k+q} - \omega_k - \omega_q} \cong \frac{i}{\omega_k + q\frac{d\omega}{dk} - \omega_k - \omega_q} = \frac{i}{qv_g - \omega_q}$$

The equation 2.3 describe the relation between action density (wave population density) $\langle N\rangle$ and the energy transfer from plasma waves to ion-acoustic waves. Since the group velocity of plasma waves is $\mathbf{v}_g = \frac{\partial\omega_k}{\partial\mathbf{k}} = \gamma_T v_{te}^2 \mathbf{k}/\omega_k > 0$, the damping of plasma waves should satisfy the condition,

$$\frac{d}{dt}\langle E_k\rangle < 0 \Rightarrow \frac{\partial\langle N\rangle}{\partial\mathbf{k}} > 0$$

i.e. the energy transfer from plasma wave quasi-particles to ion-acoustic waves requires a population inversion, and the ion-acoustic waves grow in time at the expense of plasma waves.

2.3. **Growth of ion-acoustic waves.** The influence of Langmuir waves on ion-acoustic waves is due to electron pressure from fast oscillation by the plasma waves. But the ion kinetic energy associated with this rapid oscillation is m_e/m_i times smaller than that of electrons. In slow varying scales, which is relevant to ion-acoustic waves, the rapid electron oscillation induces an radiation pressure,

$$p_{rad} = \frac{\partial}{\partial\omega}(\omega\epsilon)|_{\omega_{p0}} \frac{|E|^2}{8\pi}$$

where E is the electric field, ϵ is a dielectric function. p_{rad} is essentially plasma wave energy density. The gradient of p_{rad}, i.e. ponderomotive force, induces a slow-varying ion motion. In addition to thermal pressure $p_{th} = c_s^2 \tilde{n} m_i$, the linearized ion equation of motion is

$$m_i n_0 \frac{\partial\tilde{V}}{\partial t} = -\frac{\partial}{\partial x}(p_{th} + p_{rad})$$

plus the continuity equation,

$$\frac{\partial\tilde{n}}{\partial t} = -n_0 \frac{\partial\tilde{V}}{\partial x}$$

Wave kinetics	Envelope
phonon N	**E** envelope
Wave kinetic equation	Zakharov equation
phase space	real space
adiabatic N conserved	envelope affected by IAWs
stochastic	coherent

TABLE 1. Wave kinetics approach versus envelope formalism for Langmuir turbulence.

eliminating $\tilde{V}$, we have dynamics equation for ion-acoustic waves

$$\frac{\partial^2}{\partial t^2}\frac{\tilde{n}}{n_0} = \frac{\partial^2}{\partial x^2}\left(c_s^2\frac{\tilde{n}}{n_0} + \frac{|E|^2}{8\pi n_0 m_i}\right) \tag{2.4}$$

write $\frac{|E|^2}{8\pi n_0} = \int dk\omega_k \tilde{N}$ and use linear response of action density,

$$\tilde{N} = -\frac{q\omega_{p0}}{\Omega - qv_g}\frac{\tilde{n}}{2n_0}\frac{\partial\langle N\rangle}{\partial k}$$

then we have

$$\frac{\partial^2}{\partial t^2}\frac{\tilde{n}}{n_0} = \frac{\partial^2}{\partial x^2}\left(c_s^2\frac{\tilde{n}}{n_0} + \frac{1}{m_i}\int dk\omega_{k0}\left(-\frac{q\omega_{p0}}{\Omega - qv_g}\frac{\tilde{n}}{2n_0}\frac{\partial\langle N\rangle}{\partial k}\right)\right)$$

then the dispersion relation is

$$\begin{aligned}\Omega^2 &= q^2c_s^2 + q^2\frac{\omega_{p0}^2}{2m_i}\int dk\left(-\frac{q}{\Omega - qv_g}\frac{\partial\langle N\rangle}{\partial k}\right)\\ &= q^2c_s^2 + q^2\frac{\omega_{p0}^2}{2m_i}\int dk\left(i\pi\delta(\Omega - qv_g)\frac{\partial\langle N\rangle}{\partial k}\right)\end{aligned}$$

set $\Omega = qc_s + i\gamma_N$, $\gamma_N \ll qc_s$, then one has,

$$\Omega = qc_s + i\pi q^2\frac{\omega_{p0}^2}{4m_ic_s}\int dk\delta(\Omega - qv_g)\frac{\partial\langle N\rangle}{\partial k}$$

so the ion-acoustic wave is unstable if

$$\frac{\partial\langle N\rangle}{\partial k} > 0, \quad at\ \Omega \simeq qv_g$$

We can compare the wave kinetics approach to envelope formalism using table 1

3. Wave kinetics for zonal flow generation

It is worthwhile to note that the zonal flow growth is quite similar to the problem of Langmuir turbulence. In Langmuir turbulence, low frequency test phonons (i.e. ion-acoustic waves) grow by attracting energy from ambient plasmons (i.e. plasma waves). In this case, the zonal flow is the analogue of the ion-acoustic wave, while the drift waves are the analogue of plasma waves, and the test zonal flow interacts with a broad spectrum of drift wave fluctuations.

The essence of the theory for zonal flow growth is:

- Get mean field evolution equation of zonal flow, which relates $\partial_t\phi_{ZF}$ to $\langle\phi_{DW}^2\rangle$, in the presence of wave pressures and stresses.

- Then calculate the response of the drift wave spectrum to the test zonal flow shear.

This procedure is similar to modulational stability calculations. The time scale separation between low frequency zonal flow and high frequency drift waves enables the utilize of wave kinetics to calculate the response of the drift wave spectrum to the test zonal flow shear.

The zonal flow structure is essentially 2-dimensional. Thus in dimensionless form, the zonal flow potential evolves according to 2D vorticity equation,

$$\frac{\partial}{\partial t}\nabla_r^2\phi_{ZF} = -\frac{\partial}{\partial r}\langle\tilde{v}_r\nabla^2\tilde{\phi}_{DW}\rangle - \gamma_d\nabla_r^2\phi_{ZF}$$

i.e. this equation relates the change of zonal flow vorticity to the drift wave vorticity flux. Zonal flow evolution is then a process driven by vorticity transport, as temperature and density evolution are driven by heat and particle fluxes.

Rewrite the drift wave vorticity flux $\langle\tilde{v}_r\nabla^2\tilde{\phi}_{DW}\rangle = B\partial_r\langle\tilde{v}_r\tilde{v}_\theta\rangle$ in this equation, noting $\tilde{v}_r = -\partial_\theta\tilde{\phi}_{DW}/B$, one have,

$$\frac{\partial}{\partial t}\nabla_r^2\phi_{ZF} = \frac{1}{B}\nabla_r^2\int d^2k k_r k_\theta\left|\tilde{\phi}_k\right|^2 - \gamma_d\nabla_r^2\phi_{ZF} \tag{3.1}$$

equation 3.1 directly relates the evolution of zonal flow potential to the slow-varying envelope of the drift wave intensity.

The drift wave energy density is

$$E_k = (1 + k_\perp^2\rho_s^2)\left|\phi_k\right|^2$$

the potential enstrophy is

$$Z_k = (1 + k_\perp^2\rho_s^2)^2\left|\phi_k\right|^2$$

the drift wave dispersion relation is

$$\omega_k = \frac{\omega_{*e}}{1 + k_\perp^2\rho_s^2}$$

thus the wave action density is

$$N = \frac{E_k}{\omega_k} = (1 + k_\perp^2\rho_s^2)^2\frac{\left|\phi_k\right|^2}{\omega_{*e}}$$

the $\omega_{*e} = k_\theta V_*$ here is constant, since k_θ is unchanged by zonal flow shearing, i.e. $\frac{dk_y}{dt} = -\frac{\partial}{\partial y}(k_\theta V_{ZF}(x)) = 0$. Thus we can relate wave action density to drift wave fluctuation intensity, noting $\nabla_r^2\phi_{ZF} = iq\tilde{V}_{ZF}$, the equation 3.1 becomes

$$iq\frac{\partial}{\partial t}\tilde{V}_{ZF} = \frac{1}{B^2}\frac{\partial^2}{\partial r^2}\int d^2k\frac{k_r k_\theta}{(1 + k_\perp^2\rho_s^2)^2}\tilde{N} - \gamma_d(iq\tilde{V}_{ZF}) \tag{3.2}$$

The modulational response $\tilde{N}$ now can be calculated using linearized WKE for zonal flow shears,

$$\frac{\partial\tilde{N}}{\partial t} + v_g\frac{\partial\tilde{N}}{\partial r} + \gamma_k\tilde{N} = \frac{\partial}{\partial r}(k_\theta\tilde{V}_{ZF})\frac{\partial\langle N\rangle}{\partial k_r}$$

then the modulation $\tilde{N}$ induced by $\tilde{V}_{ZF}$ is given by

$$\tilde{N} = -\frac{qk_\theta\tilde{V}_{ZF}}{\Omega - qv_g + i\gamma_k}\frac{\partial\langle N\rangle}{\partial k_r}$$

put this back into zonal flow evolution equation, then the modulational instability eigenfrequency is,

$$\Omega = \frac{q^2}{B^2}\int d^2k \frac{k_\theta^2 k_r}{\Omega - qv_g + i\gamma_k}\frac{1}{(1+k_\perp^2\rho_s^2)^2}\frac{\partial\langle N\rangle}{\partial k_r} - i\gamma_d$$

the imaginary part gives the zonal flow growth rate

$$\Gamma = -\frac{q^2}{B^2}\int d^2k \frac{\gamma_k}{(\Omega - qv_g)^2 + \gamma_k^2}\frac{k_\theta^2 k_r}{(1+k_\perp^2\rho_s^2)^2}\frac{\partial\langle N\rangle}{\partial k_r} - \gamma_d \tag{3.3}$$

The growth of zonal flow requires $\frac{\partial\langle N\rangle}{\partial k_r} < 0$, which is satisfied for any realistic equilibrium spectrum of drift wave turbulence. In contrast to Langmuir turbulence, there is no population inversion here for zonal flow growth. This is because for drift waves are backward wave,

$$v_g = \frac{\partial\omega_k}{\partial k_r} = \frac{\partial}{\partial k_r}\frac{k_\theta V_*}{1+k_\perp^2\rho_s^2} = -\frac{2k_\theta k_r}{(1+k_\perp^2\rho_s^2)^2}V_* \Rightarrow \frac{v_g}{v_p} < 0$$

while the plasma waves are forward wave, i.e. $v_g/v_p > 0$.

On the other hand,

$$\frac{d}{dt}E_k = N\frac{d\omega_k}{dt}$$

while

$$\begin{aligned}\frac{dk_r}{dt} &= -\frac{\partial}{\partial r}(k_\theta V_{ZF}(x)) \\ &= -k_\theta V_{ZF}' \\ \frac{\partial\langle N\rangle}{\partial t} &= \frac{\partial}{\partial k_r}\left\langle k_\theta \tilde{V}_{ZF}'\tilde{N}\right\rangle\end{aligned}$$

$$\begin{aligned}\frac{d\langle E_k\rangle}{dt} &= \omega_k\frac{\partial}{\partial k_r}\left\langle k_\theta\tilde{V}_{ZF}'\tilde{N}\right\rangle \qquad (3.4)\\ &= -\sum_q\int d^2k\frac{\partial\omega_k}{\partial k_r}\frac{q^2k_\theta^2}{\Omega - qv_g + i\gamma_k}\left|\tilde{V}_{ZF}\right|^2\frac{\partial\langle N\rangle}{\partial k_r} \\ &= \frac{2}{B^2}\sum_q\int d^2k\frac{\gamma_k}{(\Omega - qv_g)^2+\gamma_k^2}\frac{q^2k_\theta^2k_r}{(1+k_\perp^2\rho_s^2)^2}\left|\tilde{V}_{ZF}\right|^2\frac{\partial\langle N\rangle}{\partial k_r} \qquad (3.5)\end{aligned}$$

since $\frac{\partial\omega_k}{\partial k_r} < 0$, growth of zonal flow by depleting energy from drift waves also requires $\frac{\partial\langle N\rangle}{\partial k_r} < 0$.

From Eq. 3.2 and 3.3, neglecting collisional damping, we have

$$\begin{aligned}\frac{d}{dt}\left|\tilde{V}_{ZF}\right|^2 &= \sum_q 2\Gamma_q\left|\tilde{V}_{ZF}\right|^2 \\ &= -\frac{2}{B^2}\sum_q\int d^2k\frac{\gamma_k}{(\Omega - qv_g)^2+\gamma_k^2}\frac{q^2k_\theta^2k_r}{(1+k_\perp^2\rho_s^2)^2}\left|\tilde{V}_{ZF}\right|^2\frac{\partial\langle N\rangle}{\partial k_r}\end{aligned}$$

It is thus apparently that

$$\frac{d}{dt}\left(\left|\tilde{V}_{ZF}\right|^2 + \langle E_k\rangle\right) = 0$$

	Plasma waves and IAWs	DWs and ZFs
High freq fluctuation	plasma wave (plasmon)	drift wave
Low freq structure	ion-acoustic wave (phonon)	zonal flow
Drive mechanism	ponderomotive pressure	Reynolds stress
wave action distribution	plasmon number	potential enstrophy $N = \left(1 + k_\perp^2 \rho_s^2\right)^2 \lvert\phi_k\rvert^2$
Modulational instability	population inversion needed	population inversion unnecessary
Regulator	ion Landau damping of phonon	collisional damping for ZFs

TABLE 2. Langmuir turbulence case versus zonal flow generation case.

so the theory conserves the energy which gives rise to a predator-prey model. Then drift wave turbulence energy is transferred into the energy of zonal flow via the modulational instability.

Now we have introduced both plasma wave-sound wave interaction (i.e. plasmon-phonon) and the drift wave-zonal flow interaction. The comparison is listed in table 2.

4. Nonlinear Schrodinger equation for Langmuir waves

4.1. Influence of ion-acoustic waves on plasma waves.

A heuristic description can be developed by applying the envelope formalism to plasmon-phonon interaction. We can write inhomogeneous plasma waves as

$$\tilde{\mathbf{E}} \sim E(x,t)\mathbf{e}_0 \exp(i\mathbf{k}\cdot\mathbf{x} - i\omega t)$$

where the $E(x,t)$ indicates the slow-varying envelope, the $\mathbf{e}_0$ denotes the polarization of the wave field, the $\exp(i\mathbf{k}\cdot\mathbf{x} - i\omega t)$ is the fast oscillating plasma wave carrier. For plasma waves, the dispersion relation is

$$\omega^2 = \omega_{pe}^2 + \alpha k^2 v_{t,e}^2$$

set $\omega = \omega_{p0} + i\gamma$ ($\gamma \ll \omega_{p0}$), plug it back, then we have

$$\begin{aligned} \omega_{p0}^2 + 2i\omega_{p0}\gamma &= \omega_{p0}^2 + \alpha v_{t.e}^2 k^2 + \omega_{p0}^2 \frac{\tilde{n}}{n_0} \\ 2i\omega_{p0}\gamma &= \alpha v_{t.e}^2 k^2 + \omega_{p0}^2 \frac{\tilde{n}}{n_0} \end{aligned}$$

since $\gamma \to \partial_t$ and $k^2 \to -\nabla^2$, we get

$$2i\omega_{p0}\frac{\partial}{\partial t}E = -\alpha v_{t,e}^2 \nabla^2 E + \omega_{p0}^2 \frac{\tilde{n}}{n_0} E \tag{4.1}$$

together with equation 2.4 which we repost here,

$$\frac{\partial^2}{\partial t^2}\frac{\tilde{n}}{n_0} = c_s^2 \nabla^2 \left(\frac{\tilde{n}}{n_0} + \frac{\lvert E\rvert^2}{8\pi n_0 m_i c_s^2} \right) \tag{4.2}$$

the set of equations 4.1 and 4.2 is knowns as Zakharov equations. They are coupled envelope equation. In the absence of nonlinear coupling, i.e. if $\tilde{n} \to 0$, Eq. 4.1 becomes the Schrodinger equation for a free particle, and if $\lvert E\rvert^2 \to 0$, Eq. 4.2 reduces to the ion-acoustic wave equation.

When the evolution of the envelope is slow, $\frac{\partial^2}{\partial t^2} \ll \frac{c_s^2}{L^2}$, the envelope modulation propagates much slower than the ion-acoustic wave. This is subsonic (adiabatic) limit. Then the ion inertial is negligible in 4.2, thus the ponderomotive force balances the gradient in thermal pressure,

$$\frac{\tilde{n}}{n_0} = -\frac{|E|^2}{8\pi n_0 c_s^2}$$

plug it this back into Eq. 4.1, the coupled Zakharov equations reduces to one nonlinear Schrodinger (NLS) equation, and is the adiabatic Zakharov equation,

$$2i\omega_{p0}\frac{\partial}{\partial t}E = -\alpha v_{t,e}^2 \nabla^2 E - \omega_{p0}^2 \frac{|E|^2}{8\pi n_0 c_s^2} E$$

The coefficient of the second term of RHS is negative which indicates an attractive potential and thus the collapse phenomenon. By introducing dimensionless variables

$$\omega_{p0} t \to t, \quad \lambda_{De}^{-1} x \to x, \quad \frac{E}{\sqrt{8\pi n_0 T_0}} \to E$$

the NLS equation is rescaled as

$$\left(i\frac{\partial}{\partial t} + \nabla^2\right) E + |E|^2 E = 0 \tag{4.3}$$

which is a cubic Schrodinger equation.

4.2. **Cubic Schrodinger equation.** In last subsection, we get the cubic Schrodinger equation as an approximation for the envelope of plasma waves affected by density perturbation (i.e. ion-acoustic waves). Here we will discuss its general significance for time-dependent dispersive waves. The general solution for a linear dispersive mode is

$$\int dk F(k) \exp\left(ikx - i\omega(k)t\right)$$

where the $\omega = \omega(k)$ is the dispersion relation. For a modulated wavetrain with most of energy at wavenumber k_0, $F(k)$ is concentrated at k_0, then the mode can be approximated by

$$\Phi = \int dk F(k) \exp\left[ikx - i\left(\omega_0 + (k-k_0)\omega_0' + \frac{1}{2}(k-k_0)^2\omega_0''\right)t\right]$$

where $\omega_0 = \omega(k_0)$, $\omega_0' = \omega_0' k_0)$ and $\omega_0'' = \omega_0''(k_0)$. This can be rewrite as

$$\Phi = \varphi \exp\left(ik_0 x - i\omega_0 t\right)$$

where

$$\varphi = \int d\kappa F(k_0 + \kappa) \exp\left[i\kappa x - i\left(\kappa\omega_0' + \frac{1}{2}\kappa^2\omega_0''\right)t\right]$$

where $k = k_0 + \kappa$ used. The φ describes the modulations, i.e. envelope evolution, it satisfies the equation

$$i\left(\partial_t\varphi + \omega_0'\partial_x\varphi\right) + \frac{1}{2}\omega_0''\partial_{xx}\varphi = 0 \tag{4.4}$$

which corresponds to the dispersion relation

$$W = \kappa\omega_0' + \frac{1}{2}\kappa^2\omega_0'' \tag{4.5}$$

The equation for Φ is

$$i\partial_t\Phi - \left(\omega_0 - k_0\omega_0' + \frac{1}{2}k_0^2\omega_0''\right)\Phi + i\left(\omega_0' - k_0\omega_0''\right)\partial_x\Phi - \frac{1}{2}\omega_0''\partial_{xx}\Phi = 0$$

corresponds to the original expansion (dispersion relation)

$$\omega = \omega_0 + (k-k_0)\omega_0' + \frac{1}{2}(k-k_0)^2\omega_0''$$

If the dispersion relation for φ is combined with a cubic nonlinearity correction, we have

$$i\left(\partial_t\varphi + \omega_0'\partial_x\varphi\right) + \frac{1}{2}\omega_0''\partial_{xx}\varphi + q\left|\varphi\right|^2\varphi = 0 \tag{4.6}$$

$\varphi = a\exp\left(i\kappa x - iWt\right)$ is still a solution, then Eq. 4.5 is modified to

$$W = \kappa\omega_0' + \frac{1}{2}\kappa^2\omega_0'' - qa^2 \tag{4.7}$$

By choosing a frame of reference moving with the linear group velocity $\omega_0^{'}$, and then rescaling the variables, Eq. 4.6 can be reduced to

$$i\partial_t E + \partial_{xx}E + \nu\left|E\right|^2 E = 0$$

it is the same form as adiabatic Zakharov equation derived in last subsection.

The one-dimensional NLS equation is known to be integrable, and the nonlinear stationary solution is soliton. These are found by looking for solutions with moving coordinate $X = x - Ut$, we set

$$E = v(X)\exp\left(irx - ist\right)$$

substituting back into NLS equation, one obtains an ordinary differential equation for v,

$$v'' + i(2r-U)v' + (s-r^2)v + \nu\left|v\right|^2 v = 0$$

then we can choose

$$r = \frac{U}{2}, \quad s = \frac{U^2}{4} - \alpha$$

and have the equation

$$v'' - \alpha v + \nu v^3 = 0$$

which can be integrated once to

$$\left(v'\right)^2 = A + \alpha v^2 - \frac{\nu}{2}v^4$$

this can be solved in elliptic functions. We set $A = 0$ and $\nu, \alpha > 0$, then the solution is

$$v = \sqrt{\frac{2\alpha}{\nu}}\operatorname{sech}\left(\sqrt{\alpha}(x-Ut)\right)$$

4.3. **Collapse of plasma waves with 3D spherical symmetry.** The Zakharov equation indicates the self-focusing of coherent plasma waves. Once the focusing starts, the local wave intensity will be increased further which leads to a singularity. The focusing also depends on the dimensionality of the system. For one-dimensional space, the the nonlinear stationary solutions of NLS equation is soliton, as shown in previous subsection. In a 3-dimensional system, focusing will continue and collapse happens.

In the system with spherical symmetry, the adiabatic Zakhrov equation becomes

$$i\frac{\partial}{\partial t}E + \frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}E + |E|^2 E = 0 \tag{4.8}$$

Being analogue to quantum mechanics, we can find out the integrals of this equation. First is the total plasmon number. If we introduce the 'flux', like the probability current in QM,

$$F = i\left(E\frac{\partial}{\partial r}E^\star - E^\star\frac{\partial}{\partial r}E\right)$$

Eq. 4.8 can be rewritten as

$$\frac{\partial}{\partial t}|E|^2 + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 F\right) = 0 \tag{4.9}$$

thus we find the total number of plasmon (i.e. total intensity of plasma waves) is conserved,

$$I_1 = \int drr^2 |E|^2$$

One the other hand, the total Hamiltonian is also conserved,

$$I_2 = \int drr^2\left(\left|\frac{\partial E}{\partial r}\right|^2 - \frac{1}{2}|E|^4\right)$$

the 1st term is kinetic energy and the 2nd potential energy. The potential is attractive, which increases as the wave amplitude increases. So the $I_2 < 0$ when wave amplitude is larger enough.

We next study the evolution of the mean square of radius

$$\langle r^2\rangle = \frac{\int drr^2r^2|E|^2}{\int drr^2|E|^2} = \frac{\int drr^4|E|^2}{I_1}$$

Noting the conservation for wave intensity Eq. 4.9, we have the relation

$$\begin{aligned}
\frac{\partial}{\partial t}\left(r^2|E|^2\right) + r^2\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2F\right) &= 0\\
\frac{\partial}{\partial t}\left(r^2|E|^2\right) + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^4F\right) - 2rF &= 0\\
\frac{\partial^2}{\partial t^2}\left(r^2|E|^2\right) + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^4\frac{\partial}{\partial t}F\right) - 2r\frac{\partial}{\partial t}F &= 0
\end{aligned}$$

averaging by $\langle \cdots \rangle = \int drr^2 (\cdots)$ then the mean square radius evolves as

$$\begin{aligned}
I_1 \frac{\partial^2}{\partial t^2} \langle r^2 \rangle &= 2 \int drr^3 \frac{\partial}{\partial t} F \\
&= 2i \int drr^3 \partial_t (E \partial_r E^* - E^* \partial_r E) \\
&= 2i \int drr^3 \left(\partial_t E \partial E_r^* - \partial_t E^* \partial_r E + E \partial_r \partial_t E^* - E^* \partial_r \partial_t E \right)
\end{aligned}$$

the RHS can be calculated by using NLS equation,

$$\begin{aligned}
i \frac{\partial}{\partial t} E + \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} E + |E|^2 E &= 0 \\
-i \frac{\partial}{\partial t} E^* + \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} E^* + |E|^2 E^* &= 0
\end{aligned}$$

$$\begin{aligned}
2i \int drr^3 \partial_t E \partial_r E^* &= -2 \int drr^3 \left(\frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} E + |E|^2 E \right) \frac{\partial E^*}{\partial r} \\
&= -2 \int dr \left[r \frac{\partial E^*}{\partial r} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} E \right) + r^3 |E|^2 E \frac{\partial E^*}{\partial r} \right] \\
&= \int dr \left[2 \frac{\partial}{\partial r} \left(r \frac{\partial E^*}{\partial r} \right) \left(r^2 \frac{\partial E}{\partial r} \right) - 2r |E|^2 E \frac{\partial E^*}{\partial r} \right] \\
&= \int dr \left[2r^2 \left| \frac{\partial E}{\partial r} \right|^2 + 2r^3 \frac{\partial E}{\partial r} \frac{\partial^2 E^*}{\partial r^2} - 2r^3 |E|^2 E \frac{\partial E^*}{\partial r} \right]
\end{aligned}$$

$$\begin{aligned}
2i \int drr^3 \partial_t E^* \partial_r E &= 2 \int drr^3 \left(\frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} E^* + |E|^2 E^* \right) \frac{\partial E}{\partial r} \\
&= 2 \int dr \left[r \frac{\partial E}{\partial r} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} E^* \right) + r^3 |E|^2 E^* \frac{\partial E}{\partial r} \right] \\
&= \int dr \left[-2 \frac{\partial}{\partial r} \left(r \frac{\partial E}{\partial r} \right) \left(r^2 \frac{\partial E^*}{\partial r} \right) + 2r^3 |E|^2 E^* \frac{\partial E}{\partial r} \right] \\
&= \int dr \left[-2r^2 \left| \frac{\partial E}{\partial r} \right|^2 - 2r^3 \frac{\partial E^*}{\partial r} \frac{\partial^2 E}{\partial r^2} + 2r^3 |E|^2 E^* \frac{\partial E}{\partial r} \right]
\end{aligned}$$

$$\begin{aligned}
2i\int drr^3E\partial_r\partial_tE^* &= 2\int drr^3E\partial_r\left(\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}E^*+|E|^2E^*\right)\\
&= -2\int dr\left(\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}E^*+|E|^2E^*\right)\frac{\partial}{\partial r}\left(r^3E\right)\\
&= -6\int dr\left(E\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}E^*\right)+r^2|E|^4\right)\\
&\quad -2\int dr\left(\frac{\partial E}{\partial r}r\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}E^*\right)+r^3|E|^2E^*\frac{\partial E}{\partial r}\right)\\
&= -6\int dr\left(-r^2\left|\frac{\partial E}{\partial r}\right|^2+r^2|E|^4\right)\\
&\quad -2\int dr\left(-\frac{\partial}{\partial r}\left(\frac{\partial E}{\partial r}r\right)\left(r^2\frac{\partial E^*}{\partial r}\right)+r^3|E|^2E^*\frac{\partial E}{\partial r}\right)\\
&= -6\int dr\left(-r^2\left|\frac{\partial E}{\partial r}\right|^2+r^2|E|^4\right)\\
&\quad -2\int dr\left(-r^2\left|\frac{\partial E}{\partial r}\right|^2-r^3\frac{\partial E}{\partial r}\frac{\partial^2E^*}{\partial r^2}+r^3|E|^2E^*\frac{\partial E}{\partial r}\right)\\
&= \int dr\left(8r^2\left|\frac{\partial E}{\partial r}\right|^2-6r^2|E|^4+2r^3\frac{\partial E}{\partial r}\frac{\partial^2E^*}{\partial r^2}-2r^3|E|^2E^*\frac{\partial E}{\partial r}\right)
\end{aligned}$$

$$\begin{aligned}
2i\int drr^3E^*\partial_r\partial_tE &= -2\int drr^3E^*\partial_r\left(\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}E+|E|^2E\right)\\
&= 2\int dr\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}E\right)+|E|^2E\right)\frac{\partial}{\partial r}\left(r^3E^*\right)\\
&= 6\int dr\left(E^*\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}E\right)+r^2|E|^4\right)\\
&\quad +2\int dr\left(\frac{\partial E^*}{\partial r}r\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}E\right)+r^3|E|^2E\frac{\partial E^*}{\partial r}\right)\\
&= 6\int dr\left(-r^2\left|\frac{\partial E}{\partial r}\right|^2+r^2|E|^4\right)\\
&\quad +2\int dr\left(-r^2\left|\frac{\partial E}{\partial r}\right|^2-3r^3\frac{\partial E}{\partial r}\frac{\partial^2E^*}{\partial r^2}+r^3|E|^2E\frac{\partial E^*}{\partial r}\right)\\
&= \int dr\left(-8r^2\left|\frac{\partial E}{\partial r}\right|^2+6r^2|E|^4-2r^3\frac{\partial E^*}{\partial r}\frac{\partial^2E}{\partial r^2}+2r^3|E|^2E\frac{\partial E^*}{\partial r}\right)
\end{aligned}$$

since

$$\begin{aligned}
\int drr^3\left(\frac{\partial E}{\partial r}\frac{\partial^2E^*}{\partial r^2}+\frac{\partial E^*}{\partial r}\frac{\partial^2E}{\partial r^2}\right) &= \int drr^3\frac{\partial}{\partial r}\left(\left|\frac{\partial E}{\partial r}\right|^2\right)\\
&= -3\int drr^2\left|\frac{\partial E}{\partial r}\right|^2
\end{aligned}$$

$$\begin{aligned}\int drr^3 |E|^2 \left(E\frac{\partial E^*}{\partial r} + E^*\frac{\partial E}{\partial r}\right) &= \int drr^3 \frac{\partial}{\partial r}|E|^2 \\ &= -3\int drr^2 |E|^2\end{aligned}$$

we finally obtain the identity

$$\begin{aligned}\frac{\partial^2}{\partial t^2}\left\langle r^2 \right\rangle &= \frac{1}{I_1}\int drr^2 \left[8\left|\frac{\partial E}{\partial r}\right|^2 - 6|E|^4 \right] \\ &= 8\frac{I_2}{I_1} - \frac{2}{I_1}\int drr^2 |E|^4\end{aligned}$$

since the second term on RHS is positive, we have

$$\frac{\partial^2}{\partial t^2}\left\langle r^2 \right\rangle < 8\frac{I_2}{I_1}$$

which gives

$$\left\langle r^2 \right\rangle = 8\frac{I_2}{I_1}t^2 + \frac{\partial}{\partial t}\left\langle r^2 \right\rangle_0 t + \left\langle r^2 \right\rangle_0$$

The integral I_1 is positive, so when I_2 is negative, the mean square radius approaches zero in finite time, i.e. the singularity forms in finite time.

5. Summary

This note presents the theory of disparate scale interaction, in the context of Langmuir turbulence, as well as the growth of zonal flow, using wave kinetic theory. The envelope formalism for Langmuir turbulence (Zakharov equations) is also presented. This problem is a fundamental paradigm for structure formation by the simplification of local symmetry-breaking perturbations by wave radiation stresses. The mechanism is often referred to 'modulational instability', since in the course of it, local modulations in the wave population field are amplified and induce structure formation. It also presents the theory on the collapse of 3D isotropic Langmuir turbulence, that is the formation of density cavity (i.e. singularity) due to the evolution of modulational instability in finite time.

The system of equations in Section 3 describes the interaction between the drift wave and zonal flow. This is an example of a 2-component, self-regulating system which leads to a predator-prey model. The drift wave turbulence grows by its own instability mechanism, and its energy is transferred to zonal flow via modulational instability process. So a correspondence is that

Drift wave fluctuation:

$$\langle N \rangle = \sum_k N_k \longleftrightarrow \textit{prey}$$

Zonal flow energy:

$$\left\langle U^2 \right\rangle = \sum_q |U_q|^2 \longleftrightarrow \textit{predator}$$

where $N_k = \left(1+\rho_s^2 k_\perp^2\right)^2 |\phi_k|^2$ and $U \equiv \frac{d}{dr}V_{ZF}$.